Formalizing Category Theory in Lean

AGMSC 2024

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Outline

Introduction

Lean

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`mathlib` has a lot of category theory.

Lean

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For the purposes of this talk, any instance of `Type` may be interpreted as Set.

(some of the following code had been modified from `mathlib` for readability.)

Categories

```
class Category (obj : Type u) : Type (max u (v + 1)) where
  id : ∀ (X : obj), (X → X)
  comp : ∀ {X Y Z : obj}, (X → Y) → (Y → Z) → (X → Z)
  id_comp : ∀ {X Y : obj} (f : X → Y), (id X) ≫ f = f
  comp_id : ∀ {X Y : obj} (f : X → Y), f ≫ (id Y) = f
  assoc : ∀ {W X Y Z : obj}
    (f : W → X) (g : X → Y) (h : Y → Z),
    (f ≫ g) ≫ h = f ≫ g ≫ h
```

Functors & Natural Transformations

```
variable (C D : Type u) [Category.{v} C] [Category.{v} D]

structure Functor : Type (max u v)
  obj : C → D
  map : ∀ {X Y : C}, (X ⟶ Y) → (obj X ⟶ obj Y)
  map_id : ∀ (X : C), map (id X) = id (obj X)
  map_comp : ∀ {X Y Z : C} (f : X ⟶ Y) (g : Y ⟶ Z),
    map (f ≫ g) = map f ≫ map g

structure NatTrans (F G : C ⥤ D) : Type (max u v) where
  component : ∀ (X : C), (F.obj X ⟶ G.obj X)
  naturality : ∀ {X Y : C} (f : X ⟶ Y),
    F.map f ≫ (component Y) = (component X) ≫ G.map f
```

The Goal

Simplicial Homotopy

I wanted to formalize the concept of a simplicial homotopy.

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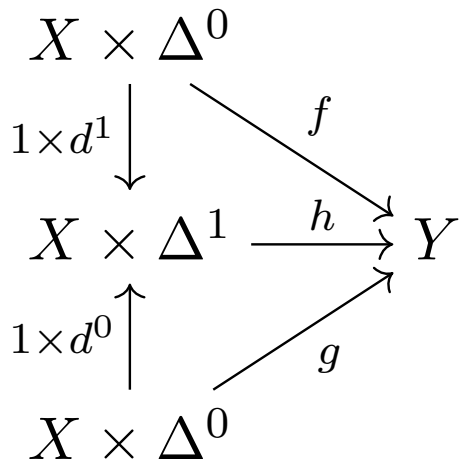
Morphisms $f, g : X \to Y$ of simplicial sets are *homotopic* if there exists some $h : X \times \Delta^1 \to Y$ such that the following diagram commutes:



i.e. "$h(x, 0) = f(x)$, $h(x, 1) = g(x)$."

Simplicial Sets

Simplicial Sets

$\mathbf{\Delta}$ is the simplex category, whose objects are finite, totally ordered sets

$$[n] = \{0, 1, \ldots, n\}$$

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A simplicial object in a category $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$.

A simplicial set is a simplicial object in **Set**. (in Lean we say it is a simplicial object in `Type`.)

The category of simplicial sets is **SSet**.

Simplicial Sets

According to `mathlib`...

```
def SimplexCategory := ℕ

def SimplicialObject (C : Type u) [Category.{v} C] :=
  SimplexCategory^op ⇒ C

def SSet : Type (u + 1) :=
  SimplicialObject (Type u)
```

The Standard Simplex

The standard n-simplex Δ^n is a simplicial set defined by the Yoneda embedding of $[n] \in \mathbf{\Delta}$:

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```
def standardSimplex : SimplexCategory ⇒ SSet :=
  yoneda
```

The Standard Simplex

For a simplicial set X and $[n] \in \mathbf{\Delta}$, we denote $X[n]$ by X_n and call this set the n-simplices of X.

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By the Yoneda lemma, X_n is in bijection with morphisms $\Delta^n \to X$:

$$\mathrm{Hom}_{\mathbf{SSet}}(\Delta^n, X) \simeq X_n$$

```
def SSet.yonedaEquiv (X : SSet) (n : SimplexCategoryᵒᵖ) :
    (Δ[n] → X) ≃ X.obj n :=
  yonedaCompUliftFunctorEquiv X n
```

Initial Formalizations

Simplicial Homotopy

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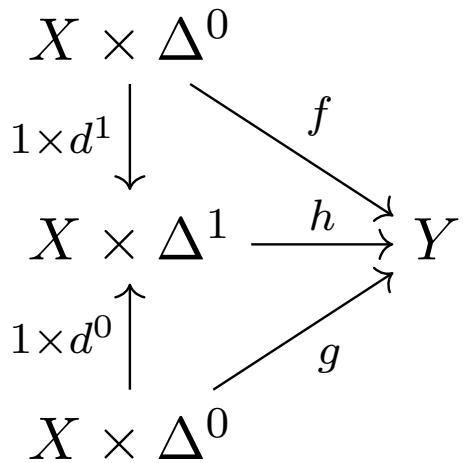
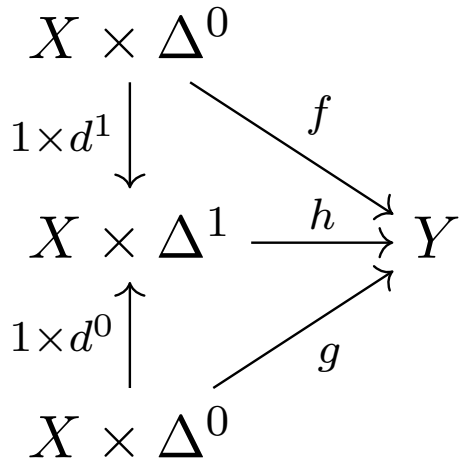


i.e. "$h(x, 0) = f(x)$, $h(x, 1) = g(x)$."

Simplicial Homotopy

At face value this is straightforward, involving only the following:

- morphisms of simplicial sets
- the standard simplex
- products of simplicial sets



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$$(X \times Y)_n = X_n \times Y_n.$$

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Problem: Lean doesn't know this.

Limits

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variable (C D : Type u) [Category.{v} C D]

/-- `C` has all (small) limits if it has limits of every
    shape that is as big as its hom-sets. -/
class HasLimits : Prop := HasLimitsOfSize.{v, v} C
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instance [HasLimits C] : HasLimits (SimplicialObject C) :=
  inferInstance
```

Products

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But it has no idea what this product looks like, because it's inherited from some super general categorical nonsense.

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This isn't a problem until you want to do explicit things with products (like we do).

Products

```
def prod (X Y : SSet) : SSet where
  obj n := X.obj n × Y.obj n
  map f n := (X.map f n.1, Y.map f n.2)

noncomputable def binaryProductIso : X × Y ≅ prod X Y :=
  limit.isoLimitCone (binaryProductLimitCone X Y)
```

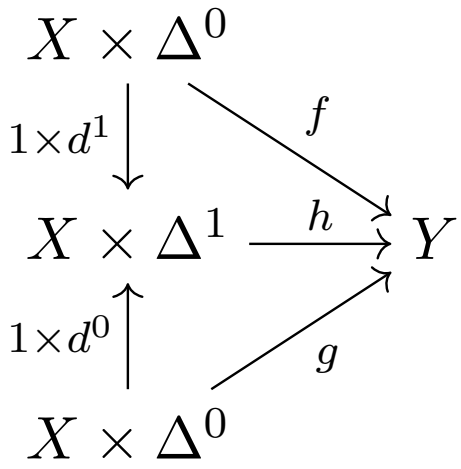
Coproducts

```
def coprod (X Y : SSet) : SSet where
  obj n := X.obj n ⨁ X.obj n
  map f n := by
    cases x with
    | inl n => exact .inl (X.map f n)
    | inr n => exact .inr (Y.map f n)

noncomputable def binaryCoproductIso : X ⨿ Y ≅ coprod X Y :=
  colimit.isoColimitCocone (binaryCoproductColimitCocone X Y)
```

Back to Homotopy

We have workable products now.



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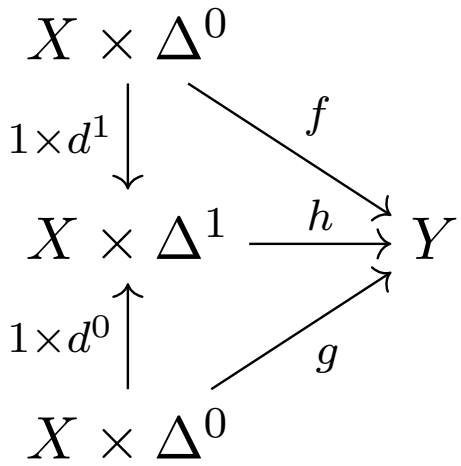

Recall that $f, g : X \to Y$, so in the diagram above we implicitly use that $X \times \Delta^0 \cong X$.

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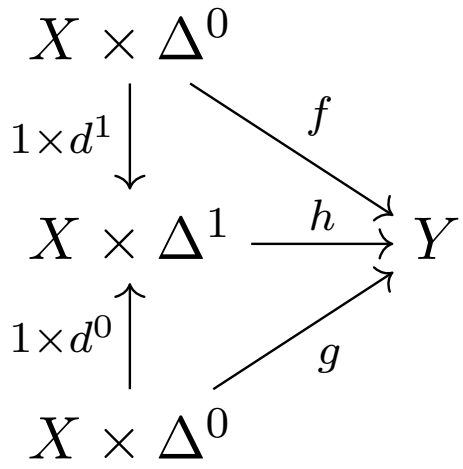

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Why does this isomorphism exist?

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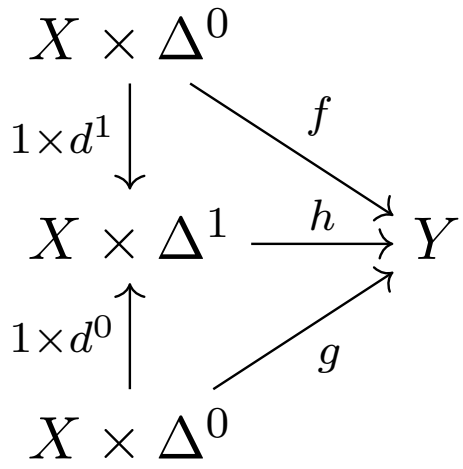

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Why does this isomorphism exist?

We get this isomorphism for free if we talk about the monoidal category structure on **SSet**.

Toward Generality

Monoidal Categories

SSet is a *monoidal category*.

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This means it has structure resembling a *monoid*: we can define a "binary operation" (called the tensor product) on simplicial sets which is associative and unital.

Monoidal Categories

```
variable (C : Type u) [Category.{v} C]

class MonoidalCategory where
  tensorProd : C → C → C -- denote by ⊗
  tensorUnit : C -- denote by 𝟙
  associator : ∀ (X Y Z : C), (X ⊗ Y) ⊗ Z ≅ X ⊗ (Y ⊗ Z)
  leftUnitor : ∀ (X : C), (tensorUnit ⊗ X) ≅ X
  rightUnitor : ∀ (X : C), (X ⊗ tensorUnit) ≅ X
  pentagon :
    /- The identity relating the isomorphism between
       `X ⊗ (Y ⊗ (Z ⊗ W))` and `((X ⊗ Y) ⊗ Z) ⊗ W` -/
  triangle :
    /- The identity relating the isomorphisms between
       `X ⊗ (𝟙 ⊗ Y)`, `(X ⊗ 𝟙) ⊗ Y`,
       and `X ⊗ Y` -/
```

Monoidal Categories

SSet has finite products, so in particular it has the empty product, which is the terminal object (in this case, Δ^0).

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Categories with finite products naturally form a *monoidal category*, with

$$X \otimes Y := X \times Y$$

and

$$\mathbb{1} := \text{the terminal object.}$$

Monoidal Categories

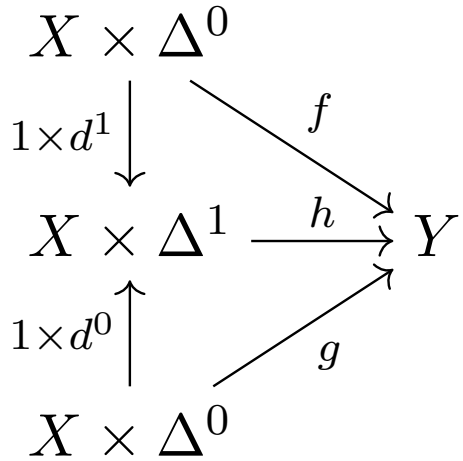
```
instance : HasFiniteProducts SSet := inferInstance

def monoidalOfHasFiniteProducts [HasFiniteProducts C] :
    MonoidalCategory C :=
  tensorProd := fun X Y ↦ X × Y
  tensorUnit := terminal C
  associator := prod.associator
  leftUnitor := fun P ↦ prod.leftUnitor P
  rightUnitor := fun P ↦ prod.rightUnitor P
```

In **SSet**, `rightUnitor` provides $X \times \Delta^0 \cong X$.

Monoidal Categories

So we defined a monoidal structure on **SSet** and now we have the desired isomorphism $X \times \Delta^0 \cong X$.



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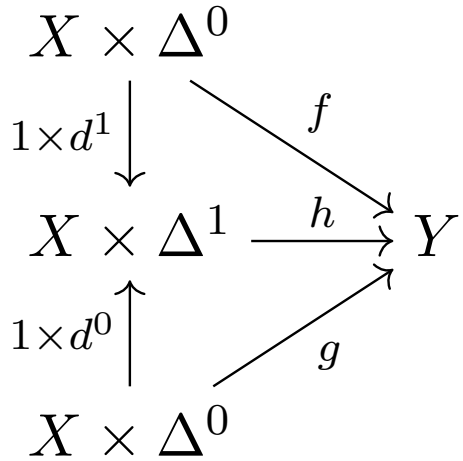

But why stop here?

More Generality

Monoidal Closed Categories

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This means that for every simplicial set Y, the functor

$$(-) \times Y : \mathbf{SSet} \to \mathbf{SSet}$$

admits a right adjoint

$$[Y, -] : \mathbf{SSet} \to \mathbf{SSet}.$$

We call $[Y, Z]$ the *internal hom* of Y and Z.

Monoidal Closed Categories

```
variable (C : Type u) [Category.{v} C]
variable [MonoidalCategory.{v} C]

class MonoidalClosed where
  /-- a choice of a right adjoint for `(-) ⊗ Y` -/
  rightAdj : (∀ Y : C), C ⇒ C
  /-- `(-) ⊗ Y` is a left adjoint -/
  adj : (∀ Y : C), tensorRight Y ⊣ rightAdj

def ihom (Y Z : C) : C ⇒ C :=
  (rightAdj Y).obj Z
```

Internal Homs

In **SSet**, the internal hom is given by

$$[Y, Z]_n := \mathrm{Hom}(\Delta^n \times Y, Z).$$

The fact that this is an adjoint follows directly from the Yoneda lemma:

$$\mathrm{Hom}(\Delta^n, [Y, Z]) \simeq [Y, Z]_n = \mathrm{Hom}(\Delta^n \times Y, Z).$$

```
def ihom (X Y : SSet) : SSet where
  obj n := Δ[n] × X → Y
  map f g := standardSimplex.map f  F ≫ g
```

Where Am I Going With This?

Given simplicial sets X and Y, we've defined a new simplicial set $[X, Y]$, the internal hom.

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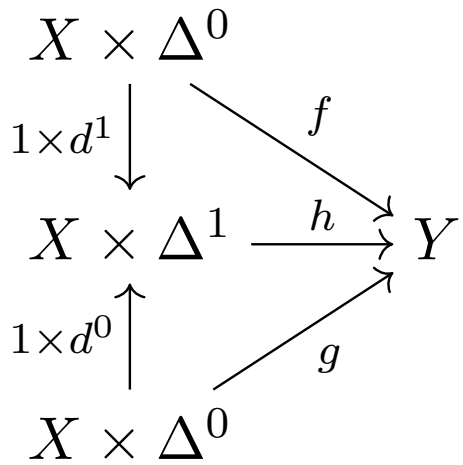
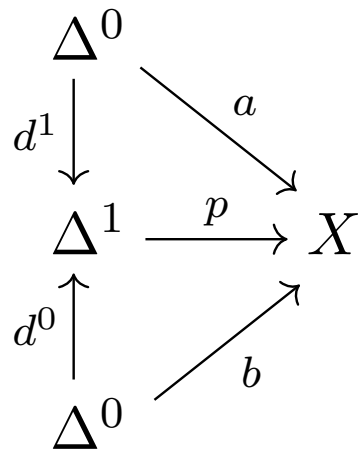
Why did we bother doing that?



Paths

A *path* in a simplicial set X is a 1-simplex $p \in X_1$. For 0-simplices $a, b \in X_0$, say p is a path from a to b if the following diagram commutes:

(recall that $X_n \simeq \mathrm{Hom}_{\mathbf{SSet}}(\Delta^n, X)$ by the Yoneda lemma.)



i.e. "$p(0) = a$, $p(1) = b$."

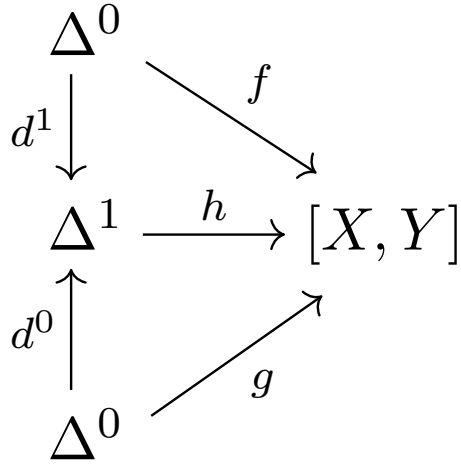
Paths

What is a path in $[X, Y]$? Let

$$h \in [X, Y]_1 = \mathrm{Hom}(\Delta^1 \times X, Y),$$

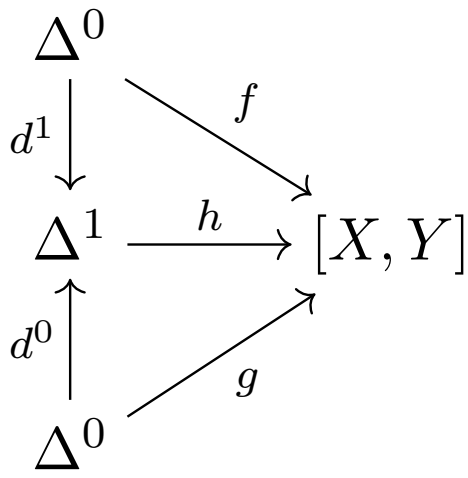
and

$$f, g \in [X, Y]_0 = \mathrm{Hom}(\Delta^0 \times X, Y) \simeq \mathrm{Hom}(X, Y).$$

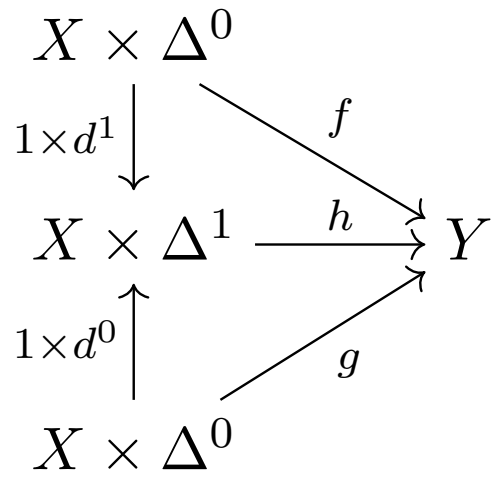


Back to Homotopy

It is then straightforward to show that a simplicial homotopy is just a path in $[X, Y]$:



path



homotopy

Conclusion

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- using this product, we defined a monoidal category structure on **SSet**
- we showed that this monoidal structure is closed
- we used the resulting internal hom to define simplicial homotopies via paths
- we built a ton of API along the way

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We can then define the notion of homotopy between morphisms in any simplicial category.

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We can then define the notion of homotopy between morphisms in any simplicial category.

SSet is a simplicial category, and in this case we recover the definition of homotopy we just came up with.

References

Emily Riehl

A Leisurely Introduction To Simplicial Sets, 2011